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Note

On the Stability of Rusanov's Third-Order Scheme*

The problem of the construction of a third order accurate finite difference scheme for a quasilinear hyperbolic system of partial differential equation was treated by Rusanov [3]. In the case of the one space dimension the Von–Neuman stability analysis was carried out and the scheme was shown to be stable. However, for the two dimensional case the stability of the scheme is not clear. It is easily shown that without a proper stabiliser the scheme is unconditionally unstable; however the form of the stabiliser has not been determined.

In this note, we would like to discuss the stability analysis of the linearized scheme in several space dimensions. We shall show that for the case of a symmetric hyperbolic system the scheme can be made stable. The form of the stabiliser and a sufficient condition for strong stability is also given. It turns out that the stabiliser is independent of the eigenvalues of the system and is very small, so that effectively the scheme is of fourth order accuracy in space.

However, for the nonsymmetric case, an example is given to show that the scheme is unconditionally unstable and there is no way to stabilize the scheme.

The extension of these results to the three dimensional case is presented in the last section.

I. The Symmetric Case

Consider the following set of equations

$$\partial u/\partial t = A(u)\,\partial u/\partial x + B(u)\,\partial u/\partial y = [\partial F(u)/\partial x] + [\partial G(u)/\partial y] \tag{1}$$

where $A(u)$, $B(u)$ are matrices, and u is a vector. Let us introduce in (x, y, t)-space the rectangular mesh with steps $\Delta x = \Delta y$ and let $\lambda = \Delta t/\Delta x$, $\phi(x_i, y_j, t_n) = \phi_{ij}^n$. Next we define the difference operators

$$\begin{aligned} \mu_x\phi &= \tfrac{1}{2}[\phi_{i+1/2,j} + \phi_{i-1/2,j}] \\ \delta_x\phi &= [\phi_{i+1/2,j} - \phi_{i-1/2,j}] \end{aligned} \tag{2}$$

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Rusanov's third order scheme can be written as

$$u^{(1)} = \mu_x\mu_y u + \frac{1}{3}\lambda(\mu_y\delta_x F + \mu_x\delta_y G)$$

$$u^{(2)} = u + \frac{2}{3}\lambda(\mu_y\delta_x F + \mu_x\delta_y G)^{(1)}$$

$$u^{n+1} = u^{(3)} = (u + \gamma_{30}(\delta_x{}^4 + \delta_y{}^4)\,u) + \frac{\lambda}{4}\left[\left(I - \frac{2}{3}\,\delta_x{}^2\right)\mu_x\delta_x F\right.$$
$$\left.\left(I - \frac{2}{3}\,\delta_y{}^2\right)\mu_y\delta_y G\right] + \frac{3}{4}\lambda(\mu_x\delta_x F_x + \mu_y\delta_y G)^{(2)} \tag{3}$$

Because of the symmetry in the space difference operators the scheme is of fourth order accuracy in space if $\gamma_{30} = 0$, but only third order in time. However in this case it is clearly unconditionally unstable, as it can be seen from Eq. (6) putting $\eta = 1$ and deriving that in this case $R = I + iT$ (where T possesses real eigenvalues). It appears γ_{30} is essential for stability.

In order to get the amplification matrix we define the following:

$$\xi = \sin\frac{\alpha}{2} \qquad \eta = \sin\frac{\beta}{2} \tag{4}$$

where α and β are the dual variables in the Fourier space. Next we define the matrix M

$$M = A\xi(1 - \eta^2)^{1/2} + B\eta(1 - \xi^2)^{1/2}. \tag{5}$$

Note that M, in the linear case, is the Fourier transform of $\frac{1}{2}(\delta_x\mu_y F + \delta_y\mu_x G)$ which is a difference approximation to $\partial u/\partial t$.

Using these notations we arrive at the following form of the amplification matrix R

$$\begin{aligned} R = I &+ 16\gamma_{30}(\xi^4 + \eta^4) + 2i(1 - \xi^2)^{1/2}(1 - \eta^2)^{1/2}\lambda M - 2(1 - \xi^2)(1 - \eta^2)\lambda^2 M^2 \\ &+ 2i[\lambda A\eta^2\xi(1 - \xi^2)^{1/2}][I + i(1 - \xi^2)^{1/2}(1 - \eta^2)^{1/2}\lambda M - \tfrac{1}{3}\lambda^2 M^2] \\ &+ 2i[\lambda B\xi^2\eta(1 - \eta^2)^{1/2}][I + i(1 - \xi^2)^{1/2}(1 - \eta^2)^{1/2} M - \tfrac{1}{3}\lambda^2 M^2] \\ &+ \tfrac{4}{3}i[\lambda A\xi^3(1 - \xi^2)^{1/2} + \lambda B\eta^3(1 - \eta^2)^{1/2}. \end{aligned} \tag{6}$$

Our aim is to find conditions on the eigenvalues of A and B such that

$$\| R^n \| \leqslant K \tag{7}$$

In this stage we assume that A and B and therefore M are symmetric. This does not imply that R is a normal matrix since A and B are not commutative, and therefore the Von Neuman condition is not sufficient. Moreover there is no way to express the eigenvalues of R in terms of those of A and B. We therefore adopt a different approach which was successful in analysing the stability of different schemes [1].

We start by introducing the following matrices

$$
\begin{aligned}
R_1 = {} & (1 - \xi^2)(1 - \eta^2) \\
& + 2i(1 - \xi^2)^{1/2} (1 - \eta^2)^{1/2} \lambda M \left\{1 - \frac{1}{4} [\xi^2(1 - \eta^2) + \eta^2(1 - \xi^2)]\right\} \\
& - 2(1 - \xi^2)(1 - \eta^2) \lambda^2 M^2 + \frac{1}{3} \lambda^2 M^2 [\xi^2(1 - \eta^2) + \eta^2(1 - \xi^2)] - \frac{4}{3} i\lambda^3 M^3 \\
R_2 = {} & \frac{1}{2} \xi^2(1 - \xi^2) + \frac{4}{3} iA\xi^3(1 - \xi^2)^{1/2} \\
R_3 = {} & \frac{1}{2} \eta^2(1 - \eta^2) + \frac{4}{3} iB\eta^3(1 - \eta^2)^{1/2} \\
R_4 = {} & \left(\frac{\eta^2(1 - \xi^2)}{2} + 2A\eta^2\xi(1 - \xi^2)^{1/2}\right)\left(I + i(1 - \xi^2)^{1/2} (1 - \eta^2)^{1/2} M - \frac{2}{3} M^2\right) \\
R_5 = {} & \left(\frac{\xi^2(1 - \eta^2)}{2} + 2B\xi^2\eta(1 - \eta^2)^{1/2}\right)\left(I + i(1 - \xi^2)^{1/2} (1 - \eta^2)^{1/2} M - \frac{2}{3} M^2\right)
\end{aligned}
\tag{8}
$$

The following theorem is easily verified.

THEOREM 1. *If $\gamma_{30} = -1/32$ then*

$$R = \sum_{i=1}^{5} R_i . \tag{9}$$

We shall try now the estimate the norms of the R_i's.

THEOREM 2. *Let $\rho(A)$, and $\rho(B)$ be the largest eigenvalues, in absolute value, of A and B respectively. Then, if*

$$\lambda\rho(A),\ \lambda\rho(B) \leqslant 1/4\sqrt{2}, \tag{10}$$

then

$$
\begin{aligned}
&\text{(a)}\quad \| R_1 \| \leqslant (1 - \xi^2)^{1/2} (1 - \eta^2)^{1/2} \Big[(1 - \xi^2)(1 - \eta^2) \\
&\qquad\qquad + \left(\frac{1}{6} + \frac{1}{64}\right)(\xi^2 + \eta^2)(\xi^2 + \eta^2 - 2\xi^2\eta^2)\Big]^{1/2} \\
&\text{(b)}\quad \| R_2 \| \leqslant \frac{1}{2} \xi^2(1 - \xi^2)^{1/2} \left[1 - \frac{7}{18} \xi^2\right] \\
&\text{(c)}\quad \| R_3 \| \leqslant \frac{1}{2} \eta^2(1 - \eta^2)^{1/2} \left[1 - \frac{7}{18} \eta^2\right] \\
&\text{(d)}\quad \| R_4 \| \leqslant \frac{1}{2} \xi^2(1 - \eta^2)^{1/2} \left(1 - \frac{\eta^2}{4}\right) \\
&\text{(e)}\quad \| R_5 \| \leqslant \frac{1}{2} \eta^2(1 - \xi^2)^{1/2} \left(1 - \frac{\xi^2}{4}\right).
\end{aligned}
\tag{11}
$$

The proofs of the inequalities (a), (b) and (c) are based on the fact that R_1, R_2, R_3 are normal since A, B and M are symmetric, therefore we have to observe only the eigenvalues of R_1, R_2 and R_3. Moreover since R_1, R_2, R_3 are each a polynomial in one matrix we can apply the spectral mapping theorem. The rest is a lengthy but straightforward algebra provided that one uses the inequality

$$\rho(M^2) = \rho^2(M) \leqslant \| M \|^2 \leqslant \max[\rho^2(A) + \rho^2(B)][\xi^2(1-\eta^2) + \eta^2(1-\xi^2)]. \quad (12)$$

For proving (d) we observe that R_4 is a product of two normal matrices. The second one is less than one in norm and the first one satisfies (d). The same argument holds for (e). This completes the proof.

Since

$$\| R \| \leqslant \sum \| R_i \| \leqslant 1 \quad (13)$$

under the condition $\lambda\rho(A)$, $\lambda\rho(B) \leqslant 1/4\sqrt{2}$, we have proved that the scheme is strongly stable under condition (10). It should be noted that this is only a sufficient condition. Computer run on model problems indicate that the actual stability limit is

$$\lambda\rho(A), \quad \lambda\rho(B) \leqslant \frac{1}{\sqrt{2}}. \quad (14)$$

Since γ_{30} is very small the scheme is effectively a fourth order scheme in space. The factor of 4 is due to the fact that our analysis yields strong stability $\| R \| \leqslant 1$ rather than $\| R^n \| \leqslant K$

II. The Nonsymmetric Case

The previous analysis uses the assumption that A, B, and M are symmetric. In case of nonsymmetric matrices the scheme might be unconditionally unstable.

Consider for example the case

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad (15)$$

and let $\xi = -\eta$. Then the amplification matrix G given in (6) is a function of $A - B$ and ξ only. But $(A - B)^n = 0$ for $n > 1$. Therefore we get the expression

$$R = \begin{bmatrix} 1 + 32\gamma_{30}\xi^4 & 2i\lambda(\xi(1-\xi^2)^{1/2} + \frac{2}{3}\xi^3(1-\xi^2)^{1/2}) \\ 0 & 1 + 32\gamma_{30}\xi^4 \end{bmatrix}. \quad (16)$$

The Buchanan criterion ([3] p. 81) states that R^n is bounded if and only if there exists a constant k_2 such that

$$| \lambda\xi(1 + \tfrac{2}{3}\xi^2)(1 - \xi^2)^{1/2} | \leqslant k_2[1 - | 1 + 32\gamma_{30}\xi^4 |] \tag{17}$$

and

$$| 1 + 32\gamma_{30}\xi^4 | \leqslant 1. \tag{18}$$

Equation (18) implies that $\gamma_{30} \leqslant 0$. Therefore for small ξ Eq. (17) is

$$\lambda | \xi |(1 + \tfrac{2}{3}\xi^2)(1 - \xi^2)^{1/2} \leqslant 32k_2 | \gamma_{30} | \xi^4 \tag{19}$$

which is not satisfied for small ξ. This indicates the fact that the scheme is unconditional unstable.

III. The Three Dimensional Case

It was observed in [1] that Rusanov's scheme is constructed by considering

$$u^{(3)} = u + \tfrac{3}{4}u_t^{(2)}(t + \tfrac{2}{3}\Delta t) + \tfrac{1}{4}\Delta t u_t(t), \tag{20}$$

where $u^{(2)}$ is a second order approximation to u.

Using this observation the third order three dimensional scheme is

$$u^{(1)} = \mu_x\mu_y\mu_z u + \frac{1}{3}\lambda[\mu_y\mu_z\delta_x F + \mu_x\mu_z\delta_y G + \mu_x\mu_y\delta_z H]$$

$$u^{(2)} = u + \frac{2}{3}\lambda[\mu_y\mu_z\delta_x F + \mu_x\mu_z\delta_y G + \mu_x\mu_y\delta_z H]^{(1)}$$

$$u^{(3)} = u - \frac{1}{32}(\delta_x^4 + \delta_y^4 + \delta_z^4)\,u + \frac{\lambda}{4}\Big\{\Big(I - \frac{2}{3}\delta_x^2\Big)\mu_x\delta_x F + \Big(I - \frac{2}{3}\delta_y^2\Big)\mu_y\delta_y G$$
$$+ \Big(I - \frac{2}{3}\delta_z^2\Big)\mu_z\mu_z H\Big\} + \frac{3}{4}\lambda\{\mu_x\delta_x F + \mu_y\delta_y G + \mu_z\delta_z H\}^{(2)}$$

For the symmetric case the same proof holds and the results are

$$\lambda\rho(A), \quad \lambda\rho(B), \quad \lambda\rho(C) \leqslant 1/4\sqrt{3}$$

Again, numerical examples indicate the fact that the factor 4 can be removed.

References

1. S. Abarbanel and D. Gottlieb, Higher order accuracy finite difference algorithms for quasi-linear conservation law hyperbolic systems, *in Math. Comp.* **27** (1973), 505–523.
2. R. D. Richtmyer and K. W. Morton, "Difference Methods for Initial Value Problems," 2nd Ed., Interscience, New York, 1967.
3. V. V. Rusanov, On difference schemes of third order accuracy for nonlinear hyperbolic systems, *J. Comp. Phys.* **5** (1970), 507–516.

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